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A NOTE ON CRITICAL REFLECTIONS OF AXIALLY SYMMETRIC WAVES IN AN ISOTROPIC, ELASTIC, HOLLOW CYLINDER†

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Abstract—At critical incidence of equivoluminal waves, the wave motion is evanescent in the case of an isotropic, elastic, hollow cylinder with traction-free cylindrical surfaces. However, it is shown that in this critical situation the wave motion can be obtained by using a suitable limiting procedure.

In a cylindrical co-opdinate system (r, θ, z), consider an isotropic, elastic, hollow cylinder, with generators parallel to the z-axis, inner radius $r = a$ outer radius $r = b$, with common center and of infinite extent otherwise. Let λ and μ be the Lamé's constants and ρ the mass density of the homogeneous, isotropic, elastic medium. In the case of axially symmetric waves, the motion of the hollow cylinder with circular cross-section is described by two Bessel equations [1]

$$\frac{\partial^2\Delta}{\partial r^2}+\frac{1}{r}\frac{\partial\Delta}{\partial r}+h^2\Delta=0, \qquad a<r<b \tag{1}$$
$$\frac{\partial^2\tilde{\omega}}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{\omega}}{\partial r}-\left(\frac{1}{r^2}-k^2\right)\tilde{\omega}=0,$$

where

$$\begin{aligned}\Delta&=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_z}{\partial z},\\ \tilde{\omega}&=\frac{1}{2}\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right),\\ h^2&=(p/v_p)^2-\gamma^2,\\ k^2&=(p/v_s)^2-\gamma^2,\\ v_p^2&=(\lambda+2\mu)/\rho,\\ v_s^2&=\mu/\rho.\end{aligned} \tag{2}$$

The wave number in the axial direction is denoted by γ, h and k are the wave numbers in the radial direction, v_p and v_s are the wave speeds of the dilatational and shear waves in an unbounded medium, and p is the circular frequency in rad. sec. For axially symmetric motion, $u_\theta \equiv 0$, $\tilde{\omega}_r \equiv 0$, $\tilde{\omega}_z \equiv 0$, $\partial\tilde{\omega}/\partial\theta \equiv (\partial\tilde{\omega}_\theta/\partial\theta) \equiv 0$ and the third equation of motion for the circumferential direction θ is satisfied identically. The two non-trivial stress components are expressed by the formulas

$$\begin{aligned}\tau_{rr}&=\lambda\Delta+2\mu\frac{\partial u_r}{\partial r},\\ \tau_{rz}&=\mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right).\end{aligned} \tag{3}$$

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We consider simple-harmonic motion of the form

$$u_r = U(r)\,\text{ixp}\,(\gamma z + pt),$$
$$u_z = W(r)\,\text{ixp}\,(\gamma z + pt), \tag{4}$$

where exp $(\ldots) \equiv \exp i(\ldots)$, and i is the fourth root of unity in the Argand plane. Then from $(2)_1$ and $(2)_2$ we find that $U(r)$ and $W(r)$ are particular solutions of the equations

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \left(\frac{1}{r^2} + \gamma^2\right) U = \left(\frac{\partial \Delta}{\partial r} + 2i\gamma\tilde{\omega}\right),$$
$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} - \gamma^2 W = \left\{i\gamma\Delta - \frac{2}{r}\frac{\partial}{\partial r}(r\tilde{\omega})\right\}, \tag{5}$$

where in these equations, and in the sequel, the exponentials have been suppressed for convenience, and Δ, $\tilde{\omega}$ are the general solutions of eqn (1).

The two ordinary differential eqns (1) are Bessel equations of integer order 0 and 1 respectively, which have a singularity as $r \to 0^+$. Therefore in the case of a solid cylinder we add the requirement that Δ and $\tilde{\omega}$ be bounded as $r \to 0^+$. In the case of hollow cylinder we take the general solution of the Bessel equations as

$$\begin{aligned} \Delta &= a_1 A_1 J_0(hr) + b_1 B_1 Y_0(hr), \quad 0 < h \\ \tilde{\omega} &= a_2 A_2 J_1(kr) + b_2 B_2 Y_1(kr), \quad 0 < k \end{aligned} \qquad a < r < b \tag{6}$$

where $a_1(h)$, $b_1(h)$ and $a_2(k)$, $b_2(k)$ are suitable functions of h and k, respectively. In the case of critical reflection when either h or k tend to 0^+, the functions a_1, b_1 or a_2, b_2 are determined by using d'Alembert's limiting procedure. The amplitude coefficients A_i, B_i $(i = 1, 2)$ are determined by the boundary conditions on the two cylinderical surfaces $r = a$ and $r = b$.

When Δ and $\tilde{\omega}$ are given by eqn (6), the non-homogeneous part of the differential eqns (5) are

$$\frac{\partial \Delta}{\partial r} + 2i\gamma\tilde{\omega} \equiv -h(a_1A_1J_1(hr) + b_1B_1Y_1(hr)) + 2i\gamma(a_2A_2J_1(kr) + b_2B_2Y_1(kr)),$$
$$i\gamma\Delta - \frac{2}{r}\frac{\partial}{\partial r}(r\tilde{\omega}) \equiv i\gamma(a_1A_1J_0(hr) + b_1B_1Y_0(hr)) - 2k(a_2A_2J_0(kr) + b_2B_2Y_0(kr)). \tag{7}$$

By using the method of *undetermined* coefficients, the particular solution of the non-homogeneous differential eqn (5) can easily be determined and is given by

$$U(r) = \frac{h}{\gamma^2 + h^2}\{a_1A_1J_1(hr) + b_1B_1Y_1(hr)\} - \frac{2i\gamma}{\gamma^2 + k^2}\{a_2A_2J_1(kr) + b_2B_2Y_1(kr)\}, \tag{8}$$
$$W(r) = \frac{-i\gamma}{\gamma^2 + h^2}\{a_1A_1J_0(hr) + b_1B_1Y_0(hr)\} + \frac{2k}{\gamma^2 + k^2}\{a_2A_2J_0(kr) + b_2B_2Y_0(kr)\},$$

where in these expressions $h > 0$ and $k > 0$. The radial wave-numbers h and k are both real when $p \geq \gamma v_p$, h becomes imaginary and k remains real when $\gamma v_s \leq p < \gamma v_p$, and both become imaginary when $p < \gamma v_s$. When $p = \gamma v_p$, $h = 0^+$ and k real; and when $p = \gamma v_s$, h is imaginary and $k = 0^+$. When $h = 0^+$, we have the case of grazing incidence of dilatational waves, and $k = 0^+$ corresponds to the case of grazing incidence of equivoluminal waves.

The coefficients a_i and b_i have still to be determined. We consider here the case of grazing incidence of equivoluminal waves when $k \to 0^+$ and h is imaginary. In this critical case, eqn $(1)_2$ degenerates into Euler's equation of second order, whose general solution is

$$\tilde{\omega} = A_2 r + B_2 \frac{1}{r}, \quad 0 < r < \infty, \quad k = 0. \tag{9}$$

It is therefore necessary that the coefficients $a_2(k)$ and $b_2(k)$ in eqn $(6)_2$ be such that

$$\text{(i)} \lim_{k\to 0^+} \{a_2(k)J_1(kr)\} = r, \qquad \text{(ii)} \lim_{k\to 0^+} \{b_2(k)Y_1(kr)\} = \frac{1}{r}. \tag{10}$$

Using d'Alembert's limiting procedure and making use of the properties of Bessel functions, we find that a_2 and b_2 are the solutions of

$$\text{(i)} \frac{d}{dk}\left(\frac{1}{a_2}\right)\bigg|_{k\to 0^+} = \frac{1}{2}, \qquad \text{(ii)} \frac{d}{dk} b_2\big|_{k\to 0^+} = -\frac{1}{2}\pi. \tag{11}$$

Therefore in the critical case of grazing incidence of equivoluminal waves when h is imaginary and $k \to 0^+$, the coefficients a_i and b_i are

$$a_1(h) = 1, b_1(h) = 1, \qquad a_2(k) = 2/k, b_2(k) = -\frac{\pi}{2}k. \tag{12}$$

One can similarly find the value of these coefficients in the case of grazing incidence of dilatation waves when $h \to 0^+$ and k remains real. However, in this brief note we will not further discuss this case, as the analytical procedure is similar to the case when $k \to 0^+$.

Therefore, a suitable solution of eqn (1), which in the limiting case contains the solution of critical reflection of equivoluminal waves in a hollow cylinder, is given by

$$\Delta = A_1J_0(hr) + B_1Y_0(hr), \quad 0 < h < i\infty, \qquad \tilde{\omega} = A_2\frac{2}{k}J_1(kr) - B_2\frac{\pi}{2}kY_1(kr), \quad 0 \le k < \infty. \tag{13}$$

The displacement components are given by

$$U(r) = \frac{h}{\gamma^2+h^2}\{A_1J_1(hr) + B_1Y_1(hr)\} - \frac{2i\gamma}{\gamma^2+k^2}\left\{A_2\frac{2}{k}J_1(kr) - B_2\frac{\pi}{2}kY_1(kr)\right\},$$
$$W(r) = \frac{-i\gamma}{\gamma^2+h^2}\{A_1J_0(hr) + B_1Y_0(hr)\} + \frac{2k}{\gamma^2+k^2}\left\{A_2\frac{2}{k}J_0(kr) - B_2\frac{\pi}{2}kY_0(kr)\right\}, \tag{14}$$

where again $0 < h < i\infty$, $0 \le k < \infty$. With Δ and $\tilde{\omega}$ given by eqn (13), we rewrite the displacement components in the convenient form

$$U(r) = -\frac{1}{(\gamma^2+h^2)}\frac{\partial\Delta}{\partial r} - \frac{2i\gamma}{\gamma^2+k^2}\tilde{\omega},$$
$$W(r) = \frac{-i\gamma}{(\gamma^2+h^2)}\Delta + \frac{1}{\gamma^2+k^2}\frac{2}{r}\frac{\partial}{\partial r}(r\tilde{\omega}). \tag{15}$$

From (3) and (15), we find that the stress components are given by

$$\tau_{rr} = \lambda\Delta + \frac{2\mu}{\gamma^2+h^2}\left(\frac{1}{r}\frac{\partial\Delta}{\partial r} + h^2\Delta\right) - \frac{4i\mu\gamma}{\gamma^2+k^2}\frac{\partial\tilde{\omega}}{\partial r},$$
$$\tau_{rz} = -2\mu\left[\frac{i\gamma}{\gamma^2+h^2}\frac{\partial\Delta}{\partial r} - \left(\frac{\gamma^2-k^2}{\gamma^2+k^2}\right)\tilde{\omega}\right]. \tag{16}$$

Substituting for Δ and $\tilde{\omega}$ from eqn (13), we find that eqn (16) takes the form

$$\begin{aligned}\tau_{rr} &= A_1\left[\lambda J_0(hr)+\frac{2\mu h}{\gamma^2+h^2}\frac{\partial}{\partial r}J_1(hr)\right]+B_1\left[\lambda Y_0(hr)+\frac{2\mu h}{\gamma^2+h^2}\frac{\partial}{\partial r}Y_1(hr)\right]\\ &\quad -\frac{4i\mu\gamma}{\gamma^2+k^2}\left[A_2\frac{2}{k}\frac{\partial}{\partial r}J_1(kr)-B_2\frac{\pi}{2}k\frac{\partial}{\partial r}Y_1(kr)\right],\\ \tau_{rz} &= \frac{2i\mu\gamma h}{\gamma^2+h^2}[A_1J_1(hr)+B_1Y_1(hr)]+2\mu\left(\frac{\gamma^2-k^2}{\gamma^2+k^2}\right)\left[A_2\frac{2}{k}J_1(kr)-B_2\frac{\pi}{2}kY_1(kr)\right],\end{aligned} \tag{17}$$

where as before $0<h<i\infty$, $0\leq k<\infty$.

In the critical case when $k=0^+$ and $0<h<i\infty$, the limiting form of stresses can easily be determined from eqn (17). Using de l'Hôpital rule, the stresses are

$$\begin{aligned}\tau_{rr}|_{k\to 0^+} &= A_1\left[\lambda J_0(h_0r)+\frac{2\mu h_0}{\gamma^2+h_0^2}\frac{\partial}{\partial r}J_1(h_0r)\right]+B_1\left[\lambda Y_0(h_0r)+\frac{2\mu h_0}{\gamma^2+h_0^2}\frac{\partial}{\partial r}Y_1(h_0r)\right]\\ &\quad -\frac{4i\mu}{\gamma}\left(A_2-B_2\frac{1}{r^2}\right),\\ \tau_{rz}|_{k\to 0^+} &= \frac{2i\mu\gamma h_0}{\gamma^2+h_0^2}[A_1J_1(h_0r)+B_1Y_1(h_0r)]+2\mu\left(A_2r+B_2\frac{1}{r}\right),\end{aligned} \tag{18}$$

where $h_0\equiv h|_{k\to 0^+}$.

When $k=0^+$, we find from $(2)_4$ that $\gamma=p/v_s$ and from $(2)_3$ we get

$$h_0^2=-(p/v_s)^2(1-\sigma^2),\quad 0<\sigma<1$$

where $\sigma\equiv v_s/v_p$. In terms of non-dimensional frequency $\Omega=p/\omega_s$, where ω_s $(=\pi v_s/2t)$ is the lowest antisymmetric thickness-shear frequency of an infinite, isotropic, elastic plate of thickness $2t$ $(=(b-a))$, we find that in the critical case of grazing incidence of equivoluminal waves

$$h_0=\frac{i\pi}{2t}\Omega(1-\sigma^2)^{1/2},\quad 0<\sigma<1 \tag{19}$$

where

$$\sigma^2\equiv\frac{1-2\nu}{2(1-\nu)},\quad \nu\equiv\frac{\lambda}{2(\lambda+\mu)},$$

and ν is the Poisson's ratio of the elastic medium.

(i) *Hollow Cylinder*

In the case of a concentric hollow cylinder of circular cross-section, vanishing of the stresses at the inner and outer surfaces gives us the frequency equation

$$\begin{vmatrix}\lambda(\gamma^2+h_0^2)J_0(h_0a)+2\mu h_0J_1'(h_0a) & \lambda(\gamma^2+h_0^2)Y_0(h_0a)+2\mu h_0Y_1'(h_0a) & \frac{4\mu}{\gamma} & -\frac{4\mu}{\gamma a^2}\\ \lambda(\gamma^2+h_0^2)J_0(h_0b)+2\mu h_0J_1'(h_0b) & \lambda(\gamma^2+h_0^2)Y_0(h_0b)+2\mu h_0Y_1'(h_0b) & \frac{4\mu}{\gamma} & -\frac{4\mu}{\gamma b^2}\\ \gamma h_0J_1(h_0a) & \gamma h_0Y_1(h_0a) & a & \frac{1}{a}\\ \gamma h_0J_1(h_0b) & \gamma h_0Y_1(h_0b) & b & \frac{1}{b}\end{vmatrix}=0, \tag{20}$$

where $J_1'(h_0a)\equiv(\partial/\partial a)J_1(h_0a),\ldots,$ and $Y_1'(h_0b)\equiv(\partial/\partial b)Y_1(h_0b)$. Using Laplacian expansion and deleting common terms in the expansion, and introducing modified Bessel functions of real argument $I_n(\theta)$ and $K_n(\theta)$, the frequency equation in the critical case of grazing incidence of equivoluminal waves is given by

$$\frac{1}{2}(1-\eta^2)\theta_a\theta_b D_{00}(\theta_a,\theta_b)+(1-\sigma^2)\theta_a\{(3+\eta^2)D_{01}(\theta_a,\theta_b)-4\eta D_{01}(\theta_a,\theta_a) \\ -4D_{01}(\theta_b,\theta_b)\}+(1-\sigma^2)\{(1+3\eta^2)\theta_b D_{10}(\theta_a,\theta_b) \\ +6(1-\eta^2)(1-\sigma^2)D_{11}(\theta_a,\theta_b)\}=0, \tag{21}$$

where

$$\begin{aligned} D_{00}(\theta_a,\theta_b)&=\{I_0(\theta_a)K_0(\theta_b)-I_0(\theta_b)K_0(\theta_a)\},\\ D_{01}(\theta_a,\theta_b)&=\{I_0(\theta_a)K_1(\theta_b)+I_1(\theta_b)K_0(\theta_a)\},\\ D_{10}(\theta_a,\theta_b)&=\{I_1(\theta_a)K_0(\theta_b)+I_0(\theta_b)K_1(\theta_a)\},\\ D_{11}(\theta_a,\theta_b)&=\{I_1(\theta_a)K_1(\theta_b)-I_1(\theta_b)K_1(\theta_a)\},\\ D_{01}(\theta_a,\theta_a)&=\{I_0(\theta_a)K_1(\theta_a)+I_1(\theta_a)K_0(\theta_a)\},\\ D_{01}(\theta_b,\theta_b)&=\{I_0(\theta_b)K_1(\theta_b)+I_1(\theta_b)K_0(\theta_b)\},\\ \theta_a&=\frac{\pi a}{2t}\Omega(1-\sigma^2)^{1/2},\\ \theta_b&=\frac{\pi b}{2t}\Omega(1-\sigma^2)^{1/2},\\ \eta&=a/b,\ a<b \text{ and } 2t=(b-a). \end{aligned} \tag{22}$$

This transcendental equation has a single root which is relatively easy to compute. In particular, finding this root of the equation does not present the problem of "spurious roots" which Gazis encountered in the investigation of a similar problem, at grazing incidence ([2] p. 274). Using asymptotic expansion of the "Hankel" type for large argument and fixed order, the frequency equation reduces to the form

$$\left(\theta_b^2+\frac{(1-\sigma^2)\kappa_1}{4\eta(1-\eta^2)}\right)\tanh(1-\eta)\theta_b-\frac{\kappa_2\theta_b}{8\eta(1-\eta)}=0,\quad O(1/\theta_b), \tag{23}$$

where

$$\begin{aligned} \kappa_1&\equiv(3+\eta^2)(1+3\eta)-(1+3\eta^2)(1+3/\eta)+48(1-\sigma^2)(1-\eta^2),\\ \kappa_2&\equiv(1-\eta)^2+16(1-\sigma^2)(1+\eta)^2. \end{aligned} \tag{24}$$

To a first approximation this equation is satisfied when

$$\theta_b\approx\frac{\kappa_2}{8\eta(1-\eta)}\left[1-16(1-\sigma^2)\eta\left(\frac{1-\eta}{1+\eta}\right)\left(\frac{\kappa_1}{\kappa_2^2}\right)\right]+\cdots O(1/\theta_b). \tag{25}$$

(ii) *Solid Cylinder*†

In the case of solid cylinder of radius $r=b$, $a\to 0$, $\eta\to 0$, $\theta_a\to 0$ and the frequency equation

†Prof. R. D. Mindlin has recently brought to our attention, that the critical reflection of waves in an isotropic, elastic plate was investigated by him in his paper, "Mathematical Theory of Vibrations of Elastic Plates", *Proceedings of the Eleventh Annual Symposium on Frequency Control*, pp. 17–40; U.S. Army Signal Corps Engineering Laboratories, Fort Monmouth, New Jersey, (1957). A remark in our earlier paper published in this journal, (1976), Vol. 12, pp. 353–357, that "Such critical reflections of elastic waves... in the case of plates,... seems to have been overlooked in the literature", was thus clearly inappropriate.

reduces to the simple form

$$\{\theta_b D_{10}(\theta_a, \theta_b) + 6(1-\sigma^2) D_{11}(\theta_a, \theta_b)\}|_{a\to 0} = 0.$$

On further simplification, we get the frequency equation

$$\theta I_0(\theta) - 6(1-\sigma^2) I_1(\theta) = 0, \tag{26}$$

where

$$\theta = \pi\Omega(1-\sigma^2)^{1/2}, \tag{27}$$

since for a solid cylinder $b = 2t$. This equation was first obtained by Onoe, *et al.*[3], as a limiting form of Pochhammer frequency equation[4], by making use of modified quotient of Bessel functions $\mathscr{L}_n(z) \equiv zJ_n(z)/J_{n+1}(z)$. This frequency equation has only one real root given by

$$\theta \approx \frac{1}{16}(47 + 5\sqrt{65}) - 3\left(1 + \frac{41}{5\sqrt{65}}\right)\sigma^2 + \ldots 0(\sigma^4). \tag{28}$$

For $\nu = 0.31$, we find that

$$\frac{\gamma b}{\delta} \equiv \xi = \frac{\theta/\delta}{(1-\sigma^2)^{1/2}} = 1.162,$$

where δ is the first root of the equation $J_1(\delta) = 0$. This is the point of intersection of the extensional branch with line OE in Fig. 2 of Ref. [3].

In this critical case, the ratio of the amplitude coefficients is

$$\left.\frac{A_1}{bA_2}\right|_{k\to 0^+} = \frac{\pi\Omega\sigma^2}{\theta I_0(\theta)}, \tag{29}$$

and the displacement components are

$$U(r)/b = \frac{ibA_2}{\pi\Omega}\left[\frac{I_1(\theta r/b)}{I_1(\theta)} - \frac{2r}{b}\right],$$
$$W(r)/b = -bA_2\left[\frac{I_0(\theta r/b)}{\theta I_1(\theta)} - \left(\frac{2}{\pi\Omega}\right)^2\right], \tag{30}$$

and $\bar{\omega} = A_2 r$.

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